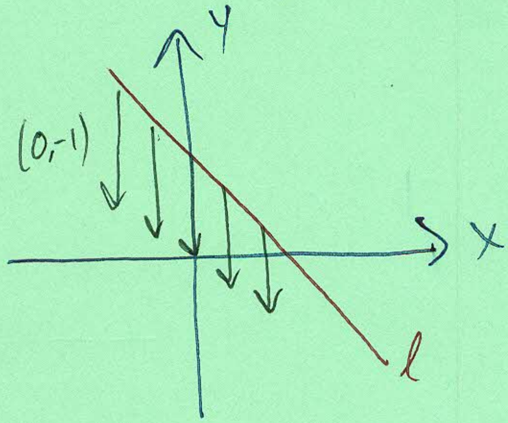
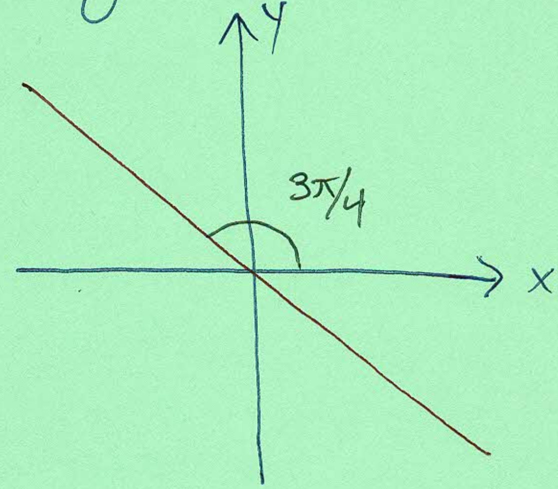
The matrices of the form A form a group called the <u>isometry group</u> of the plane.

<u>Ex</u>: Find a matrix for the reflection about the line $x+y=1$.

<u>Sol</u>: The line $x+y=1$ does not pass through the origin, so we need to first translate it to pass through the origin. The point $(0,1)$ is on the line, so if we translate by $(0,-1)$, the line will pass through the origin:



$\longrightarrow$



Now we reflect about this line, which makes an angle of $3\pi/4$ with the pos. x-axis. The final step then is to translate back, i.e., translate back by $(0,1)$

To get the matrix, we just need to multiply the matrices for the three operations:

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\frac{3\pi}{2} & \sin\frac{3\pi}{2} & 0 \\ \sin\frac{3\pi}{2} & -\cos\frac{3\pi}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

Dilations

Now, we will move away from isometries and toward more general plane transformations. We will do these for 2×2 matrices first. We begin with <u>dilations</u>, a transformation which stretches or shrinks all vectors by some factor. Suppose we want to dilate by a factor of r, that is, send a vector $\begin{pmatrix} x \\ y \end{pmatrix}$ to $\begin{pmatrix} rx \\ ry \end{pmatrix}$.

Of course, we are talking about dilations centered at the origin here.

So, if we're dilating everything by r, then

$$\hat{i} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \longrightarrow \begin{pmatrix} r \\ 0 \end{pmatrix} \quad \& \quad \hat{j} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 0 \\ r \end{pmatrix}$$

So, the matrix which dilates by a factor of r is given by

$$L_r = \begin{pmatrix} r & 0 \\ 0 & r \end{pmatrix}$$

<u>Ex</u>: Find a matrix which:

ⓐ Stretches the plane by a factor of 7.

ⓑ Shrinks the plane by a factor of 2.

<u>Sol</u>:

ⓐ $L_7 = \begin{pmatrix} 7 & 0 \\ 0 & 7 \end{pmatrix}$ ⓑ $L_{\frac{1}{2}} = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$

Even more general

We can decide how much we want to stretch in the x-direction ($\hat{\imath}$-direction) and the y-direction ($\hat{\jmath}$-direction), and we can even flip either direction by using a negative number:

Ex: Give a matrix which:

(a) Stretches by a factor of 2 in the $\hat{\imath}$-direction & shrinks by a factor of 3 in the $\hat{\jmath}$-direction.

(b) Shrinks by a factor of 4 in the $\hat{\imath}$-direction, and flips and stretches the $\hat{\jmath}$-direction by a factor of $\sqrt{3}$.

Sol:

(a) $\begin{pmatrix} 2 & 0 \\ 0 & 1/3 \end{pmatrix}$ (b) $\begin{pmatrix} 1/4 & 0 \\ 0 & -\sqrt{3} \end{pmatrix}$

Finally, we don't have to restrict ourselves to just dilating and flipping the $\hat{\imath}$ & $\hat{\jmath}$ directions. By using a trick similar to that from reflections, we can dilate and/or flip in the direction of <u>any</u> two <u>nonparallel</u> vectors

(Two vectors are parallel if $\vec{v} = c\vec{w}$ for some c.)

We just have to move them to $\hat{i}$ & $\hat{j}$, construct the matrix as before, then move $\hat{i}$ & $\hat{j}$ back.

Let's do this by example:

<u>Ex</u>: Give a matrix which stretches by a factor of 2 in the direction of $\begin{pmatrix}1\\1\end{pmatrix}$ & flips and stretches by a factor of 3 in the $\begin{pmatrix}2\\1\end{pmatrix}$ direction.

<u>Sol</u>: $\boxed{1}$ Move $\begin{pmatrix}1\\1\end{pmatrix}$ to $\hat{i}$ & $\begin{pmatrix}2\\1\end{pmatrix}$ to $\hat{j}$.

$$C = \begin{pmatrix}1 & 2\\1 & 1\end{pmatrix} \text{ takes } \hat{i} \to \begin{pmatrix}1\\1\end{pmatrix} \ \& \ \hat{j} \to \begin{pmatrix}2\\1\end{pmatrix}$$

so

$$C^{-1} = \frac{1}{1-2}\begin{pmatrix}1 & -2\\-1 & 1\end{pmatrix} = \begin{pmatrix}-1 & 2\\1 & -1\end{pmatrix} \text{ does what we need.}$$

$\boxed{2}$ Stretch $\hat{i}$ by 2, flip & stretch $\hat{j}$ by 3

$$A = \begin{pmatrix}2 & 0\\0 & -3\end{pmatrix}$$

$\boxed{3}$ Use C to move $\hat{i}$ & $\hat{j}$ back.

$$M = CAC^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 2 & -6 \\ 2 & -3 \end{pmatrix}\begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix} = \begin{pmatrix} -8 & 10 \\ -5 & 7 \end{pmatrix}$$

Combining together all of the types of matrices we have studied here (the 2×2 ones), we get the most general group of linear transformations:

<u>The General Linear Group</u>

$$GL(2;\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \middle| \ ad-bc \neq 0 \right\}$$

We can also combine these with translations to get

<u>Affine Transformation Group</u>

$$Aff(\mathbb{R}^2) = \left\{ \begin{pmatrix} a & b & x_0 \\ c & d & y_0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \text{ is in } GL(2;\mathbb{R}), \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \text{ is a vector in } \mathbb{R}^2 \right\}$$

As a final comment, what do matrices with determinant zero (i.e., $ad-bc=0$) mean?

① If $A=\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, then the entire plane collapses to the origin.

② If $A=\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, then the entire plane is flattened to the x-axis.

③ If $A=\begin{pmatrix} 2 & 4 \\ 1 & 2 \end{pmatrix}$, then the plane is collapsed onto the line in the direction of $\vec{v}=\begin{pmatrix} 2 \\ 1 \end{pmatrix}$.

All of these are examples of degenerate matrices.